Random perturbations of chaotic dynamical systems. Stability of the spectrum

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Abstract – For piecewise expanding one-dimensional maps without periodic turning points we prove that isolated eigenvalues of small (random) perturbations of these maps are close to isolated eigenvalues of the unperturbed system. (Here "eigenvalue" means eigenvalue of the corresponding Perron-Frobenius operator acting on the space of functions of bounded variation.) This result applies e.g. to the approximation of the system by a finite state Markov chain and generalizes Ulam's conjecture about the approximation of the SBR invariant measure of such a map. We provide several simple examples showing that for maps with periodic turning points and for general multidimensional smooth hyperbolic maps isolated eigenvalues are typically unstable under random perturbations. Our main tool in the 1D case is a special technique for "interchanging" the map and the perturbation, developed in our previous paper [6], combined with a compactness argument.

1 Introduction

We discuss stochastic stability in the following general framework: A discrete time dynamical system is a pair (f, X), where $X \subset \mathbb{R}^d$ is a bounded phase space (say $X = [0,1]^d$) and $f : X \to X$ is a nonsingular map, iterations of which define trajectories of the system. Nonsingular means that $m(f^{-1}A) > 0$ for any measurable set $A \subseteq X$ with positive Lebesgue measure $m(A) > 0$.

Consider now small *random perturbations* of the discrete time dynamical system. Roughly speaking this means that, when we apply f to a point $x \in X$, rather than choosing the exact value of fx we choose in a random way, in accordance with some distribution, a point from the ball $B_\varepsilon(fx)$ (i.e. with centre at the point fx and radius ε).

Definition 1.1 *Let $Q_\varepsilon(x, A)$ be a family of transition probabilities and $f : X \to X$ a map. We denote by f_ε the Markov process on the phase space X defined by the transition probabilities $Q_\varepsilon(fx, A)$ and call f_ε a* random perturbation *of f.*

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Our main assumptions on the perturbations are the following:

$$Q_\varepsilon(x, A) = 0 \quad \text{if } \mathrm{dist}(x, A) > \varepsilon \qquad \text{(locality)}, \tag{1.1}$$

$$||Q_\varepsilon h - h||_1 \le \mathbf{d}(Q_\varepsilon) \cdot \mathrm{var}(h) \to 0 \text{ as } \varepsilon \to 0 \quad \text{(smallness)}, \tag{1.2}$$

$$\mathrm{var}(Q_\varepsilon h) \le \mathrm{var}(h) + C\, ||h||_1 \qquad \text{(regularity)} \tag{1.3}$$

for any function h of bounded variation ($\mathrm{var}(h) < \infty$). The parameter ε here plays the role of a "magnitude" of the perturbation. These assumptions are satisfied for a broad class of random perturbations, among them convolutions with absolutely continuous transition probabilities, bistochastic absolutely continuous perturbations, singular perturbations of point mass type, deterministic perturbations by chaotic maps close to identity, and Ulam type perturbations (see, for example, [6] for details).

Stochastic stability of the Sinai-Bowen-Ruelle (SBR) measure of a dynamical system discussed under the same assumptions in [6] may be considered as a weak kind of stability, because other statistical characteristics may be unstable. In this paper we further explore the problem of stochastic stability and study the stability of the spectrum of the Perron-Frobenius operator $\mathbf{P}_f$ (PF-spectrum for brevity), considered as an operator on the Banach space $(\mathbf{BV}, ||\cdot||_{\mathrm{BV}})$ of functions of bounded variation. This operator describes the dynamics of densities under the action of the map f. Our main stability result provides rather general sufficient conditions for stochastic stability of isolated eigenvalues of the spectrum in the one-dimensional piecewise expanding case, and counterexamples below show that these conditions can hardly be relaxed. Moreover we demonstrate that arbitrarily small random perturbations and especially Ulam type perturbations (see below) of a generic multidimensional hyperbolic map (having a stable manifold) can completely change the PF-spectrum of the system.

One of the main motivations for the present work was to study the stability of the PF-spectrum in Ulam's construction of a finite state Markov chain approximation [15, 6] of chaotic dynamics. The idea of the construction is to take a finite partition $\{\Delta_i\}$ of the phase space with bounded volume ratios and to approximate the action of the map f by the Markov chain with transition probabilities

$$p_{ij} := \frac{|\Delta_i \cap f^{-1}\Delta_j|}{|\Delta_i|}.$$

This construction can be considered as a special type of small random perturbations, where the transition operator satisfies all above assumptions. The convergence of invariant measures of these finite Markov chains to the SBR measure of the approximating dynamical system under this construction was proved in [14] for piecewise expanding (PE) maps with large enough expanding constants, and for general PE maps in [6, 5] (see also a review there). For numerical applications of this method see e.g. [8]. Since Ulam's construction provides a rather general approach for numerical modeling of chaotic dynamics, the question of stability of the PF-spectrum and the study of nonconvolution type random perturbations becomes important not only from a purely theoretical point of view but also from a practical one.

The operator $\mathbf{P}_f$ preserves integrals, thus its leading eigenvalue is equal to 1 and the most important feature of its spectrum $\Sigma(\mathbf{P}_f)$ is the modulus of the second largest spectral value $r_2 := \sup\{|\tau| : \tau \in \Sigma(\mathbf{P}_f),\ |\tau| < 1\}$, which characterizes the rate of convergence to the SBR measure. The corresponding value for the perturbed operator is denoted by $\tilde{r}_2$.

The basic idea here is the following. Recall that $||h||_{\text{BV}} = \text{var}(h) + ||h||_1$. In [6] we proved that the transition operator $P_\varepsilon = Q_\varepsilon \mathbf{P}_f$ of the randomly perturbed map satisfies the uniform Lasota-Yorke type inequality

$$\left|\left|P_\varepsilon^N h\right|\right|_{\text{BV}} \leq \alpha \cdot ||h||_{\text{BV}} + C \cdot ||h||_1 \tag{1.4}$$

for $h \in \mathbf{BV}$, some fixed integer N, $\alpha \in (0,1)$ and $C > 0$ independent of ε. This yields at once the existence of f_ε-invariant densities h_ε with $||h_\varepsilon||_{\text{BV}} \leq \frac{C}{1-\alpha}$, such that (1.2) forces each weak limit $h_* = \lim_{\varepsilon \to 0} h_\varepsilon$ to be an invariant density for f. (In fact, as $||h_\varepsilon||_{\text{BV}}$ is uniformly bounded, h_* is not only a weak limit, but $||h_* - h_\varepsilon||_1 \to 0$ as $\varepsilon \to 0$.) If the expanding constant λ is larger than 2, (1.4) was proved with $N = 1$ in various settings, see e.g. [10, 3, 12] and references therein. For quite a while it was supposed that the extension of this inequality to cases with $\lambda \in (1, 2]$ is only a technical problem. However, the counterexample constructed in [4] shows that the situation is not so simple. After this counterexample it became clear, that the main problem is the possible existence of periodic *turning* points, i.e. points where the derivative of the map f is not well defined. Namely, under the action of random perturbations "traps" or "absorbing sets" can appear near these periodic turning points, which leads to the appearance of new localized ergodic components in the perturbed system.

Definition 1.2 *Let $X = [0, 1]$. A map $f : X \to X$ is* piecewise C^2 *if there exists a partition of X into disjoint intervals $\{X_j\}$, such that $f|_{\text{Clos}(X_j)}$ is a C^2-diffeomorphism (of the closed interval* $\text{Clos}(X_j)$ *to its image). Its* expanding constant *is defined as*

$$\lambda_f := \inf_{j,\, x \in X_j} |f'(x)|.$$

A piecewise C^2 map is called piecewise expanding *(PE), if $\lambda_{f^k} > 1$ for some iterate f^k.*

Definition 1.3 *The* image *of a measure μ under the action of a map f is the measure $f\mu$ defined by $f\mu(A) = \mu(f^{-1}A)$ for any measurable set A. By $f_\varepsilon\mu$ we mean the measure $f_\varepsilon\mu(A) = \int Q_\varepsilon(fx, A) d\mu(x)$. A measure μ is f (f_ε)–invariant, if $f\mu = \mu$ ($f_\varepsilon\mu = \mu$). μ is called* smooth, *if it has a density with respect to Lebesgue measure.*

Definition 1.4 *A* turning point *of a map f is a point, where the derivative of the map is not well defined. We denote the set of turning points by* TP, *and the set of periodic turning points by* PTP.

Recall that the *essential spectral radius* of the operator $\mathbf{P}_f$ is the smallest nonnegative number θ for which elements of the spectrum $\Sigma(\mathbf{P}_f)$ outside of the disk of radius θ centered at the origin are isolated eigenvalues of finite multiplicity. In [11] it was shown that for a PE map $\theta = \lim_{n \to \infty} \sqrt[n]{\lambda_{f^n}^{-1}}$. Therefore f is PE if and only if $\theta < 1$. From now on we fix some numbers $\theta', \tilde{\theta}$ arbitrarily close to θ with $\theta < \theta' < \tilde{\theta} < 1$.

Theorem 1.1 *Let f be a piecewise expanding map with* $\text{PTP} = \emptyset$, *and let perturbations Q_ε satisfy (1.1), (1.2) and (1.3). Let r be an accumulation point of eigenvalues $r_\varepsilon > \tilde{\theta} > \theta$ of the perturbed operators $P_\varepsilon := Q_\varepsilon \mathbf{P}_f$ for $\varepsilon \to 0$. Then there are a sequence $k \to \infty$ and a function $h \in BV$ such that $r_{\varepsilon_k} \to r$ and $||h_{\varepsilon_k} - h||_1 \to 0$ as $k \to \infty$, where $\mathbf{P}_f h = rh$ and $P_{\varepsilon_k} h_{\varepsilon_k} = r_{\varepsilon_k} h_{\varepsilon_k}$.*

In other words, any accumulation point of the eigenvalues of the perturbed operators lying outside of the disk containing the essential spectrum of the original operator is an isolated eigenvalue of $\mathbf{P}_f$. This generalizes the corresponding part of the results for convolution type perturbations of smooth expanding maps in [2]. In the case of Ulam type approximations based on Markov partitions of smooth hyperbolic maps our theorem is complemented by the following result [9]: Each isolated eigenvalue is a limit point of eigenvalues of the corresponding Ulam operators.

It is worth to remark that the rate of convergence of the eigenfunctions for the eigenvalue 1 is $O(\varepsilon|\log\varepsilon|)$ whereas we have no rates of convergence for other spectral quantities. In particular, in some cases numerical experiments show very slow convergence of the r_ε to true eigenvalues.

Technically the proof of this theorem is based on the following proposition, being a consequence of several technical results obtained in [6].

Proposition 1.1 *Let f be a piecewise expanding map with* $\mathrm{PTP} = \emptyset$, *and let perturbations Q_ε satisfy (1.1), (1.2) and (1.3). Then there exist constants C, ε_0 such that for some finite N*

$$\mathrm{var}(P_\varepsilon^N h) \leq \tilde{\theta}^N \mathrm{var}(h) + C\,||h||_1$$

for any $\varepsilon \in (0, \varepsilon_0)$ and any function $h \in \mathbf{BV}$.

The paper is organized as follows. In Section 2 we recall some necessary definitions and prove our main stability result. Section 3 is devoted to the analysis of various situations when the spectrum is not stable with respect to random perturbations. Especially important among these situations are the spectrum collapse in the absence of isolated eigenvalues and the instability of the spectrum of multidimensional hyperbolic maps due to the presence of the stable foliation of the map. Finally we discuss a possible generalization of the notion of the spectrum by means of zero-noise limit of the spectra of randomly perturbed systems.

2 Proof of the stability result

Recall that the variation of a function over $h : X = [0,1] \to \mathbb{R}_1$ is defined as $\mathrm{var}(h) := \sup\{\int_X \phi' h\,dx\}$, where the supremum is taken over all functions $\phi \in C^1(\mathbb{R})$ with compact support, $||\phi||_\infty \leq 1$ and $||\phi'||_\infty < \infty$. Notice that as $X = [0,1]$ is bounded, $||h||_1 \leq \frac{1}{2}\mathrm{var}(h)$ for all $h \in \mathbf{BV}$, and if $I \subseteq X$ is an interval, then $\mathrm{var}(h \cdot \mathbf{1}_I) \leq \mathrm{var}(h)$. Indeed, our setting means that $\mathrm{var}(h)$ is the variation of $h \cdot \mathbf{1}_X$ over $\mathbb{R}$.

Given N and $\beta > 0$ as defined above we refine the partition $\mathcal{Z}$ into intervals of monotonicity of the map f by adding further points to TP in such a way that

$$\mathrm{var}\left(\lambda_{f|Z}^{-1} f'_{|Z}\right),\ \mathrm{var}\left((\lambda_{f|Z} f'_{|Z})^{-1}\right) \leq \beta\ , \tag{2.1}$$

not introducing new TP that are mapped to other TP.

Define $\tilde{P}_1, \tilde{P}_2 : \mathbf{BV} \to \mathbf{BV}$ by

$$\tilde{P}_1 h = Q_\varepsilon \mathbf{P}_f(h \cdot \mathbf{1}_{X\setminus Y}), \quad \tilde{P}_2 h = Q_\varepsilon \mathbf{P}_f(h \cdot \mathbf{1}_Y)$$

where Y is a neighbourhood of TP scaling linearly with respect to ε (see [6]).

Proposition 2.1 *[6, Proposition 3.1] Suppose that there are constants $C_1, C_2 > 0$ and $\alpha \in (0,1)$ such that*

$$\mathrm{var}(\tilde{P}_j^k h) \leq C_1 \alpha^k \mathrm{var}(h) + C_2 \, ||h||_1 \quad \textit{for all } k \in \mathbb{Z}_+ \tag{2.2}$$

and that there is some $N \in \mathbb{Z}_+$ such that

$$\tilde{P}_2 \tilde{P}_1^k \tilde{P}_2 = 0 \quad \textit{for all } k = 1, \ldots, N. \tag{2.3}$$

Then

$$\begin{aligned} \mathrm{var}((Q_\varepsilon \mathbf{P})^N h) \leq & \left(\frac{N(N+1)}{2} C_1^3 + C_1 \right) \alpha^N \cdot \mathrm{var}(h) \\ & + (1 + C_1 + C_1^2) C_2 \frac{(N+1)^2}{2} \cdot ||h||_1 \ . \end{aligned}$$

The assumptions (2.2) and (2.3) were verified in [6] for PE maps with PTP $= \emptyset$ and for sufficiently small ε without paying attention to a particularly sharp estimate of the constant α. For the purposes of our present paper we need to show that α can be chosen as $\alpha = \theta'$, i.e. close to the essential spectral radius θ.

Since PTP $= \emptyset$, there is some n_0 such that $\tilde{P}_2^{n_0} = 0$ so that inequality (2.2) can be satisfied for any positive α with suitable constants C_1, C_2 depending only on f and Q. For $\tilde{P}_1$ the relevant estimate is given in [6, Proposition 3.8]. The value $\alpha = (\frac{3}{4})^{1/N}$ given in the statement of the proposition is not as sharp as the corresponding proof permits. In fact, the effictive estimate derived in the proof is

$$\begin{aligned} & \mathrm{var}(\tilde{P}_1^N h) \\ & \leq (1 + \frac{3}{2}\beta)^{2N} (\lambda_{f^N}^{-1} + \frac{1}{2} N \beta \lambda_f^{-N}) \mathrm{var}(h) + (\lambda_{f^N}^{-1} + \frac{1}{2} N \beta \lambda_f^{-N}) \tilde{C}_N \cdot ||h||_1 \end{aligned}$$

for each $h \in \mathbf{BV}$. [1]

Proof of Proposition 1.1. It remains to choose the constant $\beta > 0$ so small that

$$(1 + \frac{3}{2}\beta)^{2N} (\theta^N + \frac{1}{2} N \beta \lambda_f^{-N}) < (\theta')^N < \tilde{\theta}^N$$

to obtain the statement of Proposition 1.1. ■

Proof of Theorem 1.1. Suppose that $P_\varepsilon h_\varepsilon = r_\varepsilon h_\varepsilon$ for some $h_\varepsilon \in BV$ with $||h_\varepsilon||_1 = 1$ and $|r_\varepsilon| \geq \theta'$. Then there is a constant $S > 0$ depending only on $\tilde{\theta}$ and on the constants in the Lasota-Yorke type inequality such that $\mathrm{var}(h_\varepsilon) \leq S$. Indeed, by Proposition 1.1

$$\mathrm{var}(P_\varepsilon^N h_\varepsilon) \leq \tilde{\theta}^N \mathrm{var}(h_\varepsilon) + C \, ||h_\varepsilon||_1 \, .$$

Since $P_\varepsilon^N h_\varepsilon = r_\varepsilon^N h_\varepsilon$, we obtain the following estimate:

$$\mathrm{var}(h_\varepsilon) = \mathrm{var}(\frac{1}{r_\varepsilon^N} P_\varepsilon^N h_\varepsilon) \leq \left(\frac{\tilde{\theta}}{r_\varepsilon} \right)^N \mathrm{var}(h_\varepsilon) + \frac{C}{r_\varepsilon^N} \, ||h_\varepsilon||_1 \, ,$$

[1] Observe that the brackets in the corresponding formula in [6] are set slightly wrong, which had no further effect on the subsequent proofs in that paper.

or

$$\operatorname{var}(h_\varepsilon) \le \frac{C}{r_\varepsilon^N - \tilde{\theta}^N} := S < \infty,$$

since $r_\varepsilon > \tilde{\theta}$. In particular also $\operatorname{var}(\mathbf{P}_f h_\varepsilon) \le S$ and $\operatorname{var}(\mathbf{P}_f h) \le S$, where $h = h_0$.

Let r be any accumulation point of the r_ε for $\varepsilon \to 0$. Then there are a sequence $\varepsilon_k \to 0$ and a function $h \in BV$ such that $r_{\varepsilon_k} \to r$ and $||h_{\varepsilon_k} - h||_1 \to 0$ as $k \to \infty$. In particular $\operatorname{var}(h) \le S$. Using the assumption (1.2) it follows that

$$\begin{aligned} ||\mathbf{P}_f h - P_{\varepsilon_k} h_{\varepsilon_k}||_1 &\le ||(\mathbf{P}_f - P_{\varepsilon_k})h||_1 + ||P_{\varepsilon_k}(h - h_{\varepsilon_k})||_1 \\ &\le \mathbf{d}(Q_{\varepsilon_k}) \cdot S + ||h - h_{\varepsilon_k}||_1 \to 0 \end{aligned}$$

as $\varepsilon \to 0$ and

$$||rh - r_{\varepsilon_k} h_{\varepsilon_k}||_1 \le |r| \cdot ||h - h_{\varepsilon_k}||_1 + |r - r_{\varepsilon_k}| \cdot ||h_{\varepsilon_k}||_1 \to 0 \ .$$

Hence $\mathbf{P}_f h = rh$. ■

3 Instability of the PF-spectrum for generic maps

3.1 Instability of the essential spectrum

The above estimates ensure stochastic stability of the isolated eigenvalues of $\mathbf{P}_f$ in the sense that spectral points outside the essential spectrum cannot arise "from nothing" under perturbations. We have no examples of maps f without PTP and perturbations Q_ε where an isolated eigenvalue of $\mathbf{P}_f$ is not approximated by eigenvalues of the P_ε. However, if the only isolated eigenvalue is 1, the essential spectrum may collapse, which leads to spectrum localization ($\tilde{r}_2 = 0$). The simplest example when this phenomenon takes place is the case of Ulam perturbations of the well known dyadic map $x \to 2x \,(\mathrm{mod}\ 1)$, when the number of intervals in the Ulam partition is equal to 2^n. If this number is not an integer power of 2 there will be a stable eigenvalue $\tilde{r}_2 = 1/2$ (on the boundary of the essential spectrum) with the eigenvector $(x - 1/2)$. The following statement generalizes this observation.

Theorem 3.1 *Let $f : X \to X \subset \mathbb{R}^d$ be piecewise linear with respect to a partition $\mathcal{Z}$ and assume that $f(I) = X$ for each $I \in \mathcal{Z}$. Denote by $\mathcal{Z}_N$ the partition of X into domains of linearity of f^N. Then the spectrum of the Ulam operator $P_{\mathcal{Z}_N}$ constructed with respect to the partition $\mathcal{Z}_N$ consists of only 0 and 1.*

Sketch of **proof**: Let h be a piecewise constant function on $\mathcal{Z}_N$ such that $\int h(x)\,dx = 0$. Then $P_{\mathcal{Z}_N}^N h = 0$, which proves that $\tilde{r}_2 = 0$.

3.2 Instability in the periodic turning points case

In this more general case we cannot proceed as above in spite of the fact that the first eigenvector is stable under some additional restrictions (see [6] for details). The reason is that the proof of this statement in [6] uses slightly weaker estimates of the rate of convergence, which do not ensure our Proposition 1.1. In this section we want to argue that isolated

eigenvalues of the spectrum (except 1) might be unstable indeed. To show this we modify the W-map (described in detail in [6]) such that locally around the fixed turning point c it remains the same, while no point "outside" is mapped into a small neighborhood of the point c. The modified W-map is shown in the Figure 1. Observe that this map is Markov. Consider the following random perturbation: $x \to x+\varepsilon$ with probability $1-\delta$ and $x \to x$ with probability δ. We apply this random perturbation only in the neighborhood of the point c consisting of two neighbouring intervals of monotonicity. For $\delta = 0$ there is a ε-small ergodic component I_ε around the point c. For small enough $\varepsilon, \delta > 0$ the escape rate out of the set I_ε is of order δ. By the construction of the map, the probability to hit into the interval I_ε starting outside of it is equal to zero. Therefore the transition operator corresponding to the randomly perturbed system must have an eigenvalue of order $1 - \delta$. Thus the spectral gap of the perturbed operator vanishes as $\delta \to 0$.

Example 1. Let us discuss now in more detail Ulam perturbations of the modified W-map. Let x be a local coordinate such that the fixed turning point be 0, and denote the Ulam interval containing this point by $I = [-a, b]$. Define $\lambda = |f'_{|_I}|$. Then the probability to remain in the inteval $I = [-a, b]$ in Ulam's approximation is equal to

$$p(a, b) = \frac{a/\lambda + \min\{a, \lambda b\}/\lambda}{a+b} = \frac{a + \min\{a, \lambda b\}}{\lambda(a+b)}.$$

Let $a = \lambda b$. Then

$$p(\lambda b, b) = \frac{\lambda b + \lambda b}{\lambda(\lambda b + b)} = \frac{2}{\lambda+1} = 1 - \frac{\lambda - 1}{\lambda + 1}.$$

On the other hand, if $a \le b$ then the image of any other Ulam interval does not intersect with the interval I. This shows that $2/(\lambda+1) < 1$ belongs to the spectrum of the perturbed operator, whereas the special form of the modified W-map implies that the spectrum of $\mathbf{P}_f$, except for the eigenvalue 1, is contained in the circle with radius $1/\lambda < 2/(\lambda+1)$. Observe that in this case the perturbation cannot make the spectral gap arbitrarily small, but r_2 might be unstable. In any case this proves the appearance of an eigenvalue of the perturbed operator in the "spectral gap" of the unperturbed one.

One might argue that still it is possible that under the action of well behaving random perturbations of convolution type the influence of just a few periodic turning points may not matter. To show that this is not the case, consider the following symmetric random perturbation:

$$x \to \begin{cases} x - \varepsilon & \text{with probability } q \\ x & \text{with probability } 1 - 2q \\ x + \varepsilon & \text{with probability } q, \end{cases}$$

with $0 < q \ll 1$. We apply this pertutbation to the same modified W-map for $\lambda = 2$ (notice, that in this case $r_2 = 1/2$), and show that the corresponding transition oprator $Q_\varepsilon \mathbf{P}$ has an eigenvalue $1 - 2q > 1/\lambda$. Since no point outside of a small neighborhood of the fixed point can hit into this neighborhood, it is enough (as in our previous examples) to study the escape rate from this neighborhood. If $\lambda = 2$ then locally the behaviour of the randomly perturbed system is completely described by the following random walk model on $\mathbb{Z}$:

$$x \to \begin{cases} 2x + \xi & \text{if } x \ge 0 \\ -2x + \xi & \text{otherwise,} \end{cases}$$

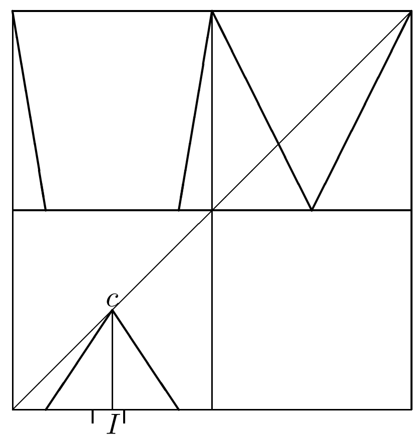

Figure 1: The modified W-map.

$$\xi = \begin{cases} -1 & \text{with probability } q \\ 0 & \text{with probability } 1-2q \\ 1 & \text{with probability } q. \end{cases}$$

Clearly, if $|x| > 1$ then the trajectory of this point will never return to zero. Consider the part of the transition matrix corresponding to the points -1, 0 and 1. It can be written as

$$\begin{pmatrix} q & 0 & 0 \\ q & 1-2q & q \\ q & 0 & 0 \end{pmatrix}$$

Thus this matrix has an eigenvalue $1-2q$, which proves our statement.

3.3 Instability for a generic multidimensional hyperbolic map

Stability of spectral properties becomes a much more delicate problem in the multidimensional case. Traditionally, to define this spectrum one considers the Perron-Frobenius operator for the expanding map defined on unstable manifolds induced by the original map (see, for example, [16, 9, 7]). Another way to calculate the isolated eigenvalues is to study the so called weighted dynamical ζ-function of a map, which counts periodic points of the map weighted by Jacobians in the unstable direction. Zeros and poles of the ζ-function correspond to isolated eigenvalues of the map (see, for example, [1] and references therein).

Our spectral stability results can be generalized for finite systems of weakly coupled 1D PE maps (see [5] for definitions). On the other hand, even for a more general multidimensional PE map our construction does not work, since there is no good control over coefficients of a Lasota-Yorke type inequality in this case, contrary to the 1D case.

Example 2. Consider a smooth hyperbolic map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Let 0 be a hyperbolic fixed point of the map, and let the horizontal direction be locally unstable with the expanding constant $\lambda_u > 1$, while the vertical direction be contracting with $\lambda_s \ll 1$. We consider Ulam partitions into equal squares rotated by the angle $\pi/2$ with respect to the coordinate axes.

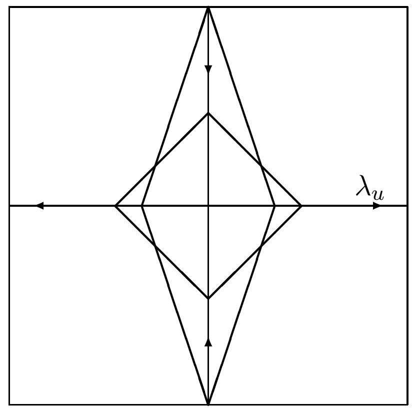

Figure 2: One element of the rotated Ulam partition and its preimage in the 2D hyperbolic case.

One element of the partition together with its preimage is shown in Figure 2. Straightforward calculations show that the probability to remain in the considered Ulam square is of order

$$p \overset{(\lambda_s \ll 1)}{\approx} 1 - \left(1 - \frac{1}{\lambda_u}\right)^2 = \frac{1}{\lambda_u}\left(2 - \frac{1}{\lambda_u}\right) = \frac{3}{4}|_{\lambda_u = 2}.$$

One can calculate this probability exactly also for the case of finite values of λ_s, for example $p = 2/3$ for $\lambda_s = 1/2$. This is not sufficient to prove that $\tilde{r}_2 > r_2$, but it indicates that the limit behaviour of the approximation might differ from that of the original map. A slightly less striking example of this type was discussed in [13], where it was claimed that the worst situation is when the angle between the contracting and expanding directions is small.

The following numerical example shows that the untypical instability of the essential spectrum due to the presence of periodic turning points in the one-dimensional case becomes typical for multidimensional maps. Near a periodic point of a multidimensional hyperbolic map stable and unstable foliations are coming arbitrarily close one to another. Therefore an arbitrarily small (random) perturbation can mix them (similarly to the situation near periodic turning points (see [6])). We study an example as simple as possible to demonstrate that this type of behavior is generic.

Example 3. Consider the well known “cat” map, which is the simplest example of a smooth two dimensional hyperbolic map. This is a map from the unit torus $X = [0,1] \times [0,1]$ into itself defined by $(x, y) \mapsto (x + y \,(\mathrm{mod}\ 1)\,, x + 2y \,(\mathrm{mod}\ 1)\,)$. We consider two partitions of X into equal squares. First we simply divide horizontal and vertical axes into n equal intevals, whose products give a partition into $N = n^2$ squares, which we call the standard partition. There is a one to one correspondence of these squares and pairs of integers ($i = nx, j = ny$), where (x, y) is the pair of coordinates of the lower left corner of a square. Here $i, j \in \{0, 1, \ldots, n-1\}$. Simple calculation gives the following transition probabilities for the corresponding Markov chain whose elements are numbered as $jn + i + 1$ (see also Figure 3):

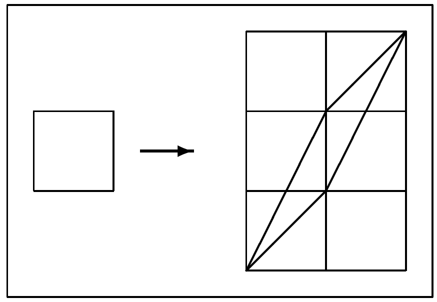

Figure 3: Image of an element of the "standard" partition by the "cat" map.

	$i+j, i+2j$	$i+j, i+2j+1$	$i+j+1, i+2j+1$	$i+j+1, i+2j+2$
i, j	1/4	1/4	1/4	1/4

Moduli of the "second" eigenvalues (r_2) of the transition matrices ($n^2 \times n^2$), and their multiplicities (in parenthesis) are shown in the following table:

n	r_2	n	r_2	n	r_2	n	r_2	n	r_2
2	0.0000(3)	3	0.3536(8)	4	0.0000(15)	5	0.3299(20)	6	0.3536(8)
7	0.4886(16)	8	0.4454(24)	9	0.3847(24)	10	0.4275(12)	11	0.5161(20)
12	0.3783(48)	13	0.4835(28)	14	0.4886(16)	15	0.4045(16)	16	0.4454(24)
17	0.4335(24)	18	0.5957(24)	19	0.5387(36)	20	0.4275(12)	21	0.5357(32)

Compare this with the inverse to the largest eigenvalue of our linear map $1/\Lambda = 2/(3+\sqrt{5}) \approx 0.38204$, which (see, for example, [7]) is the correct value of the "second" eigenvalue in this case.

To show that even this is not the worst case we consider also another partition, namely the standard partition shifted by $1/(2n)$ (in both directions). Similarly to the previous case, we associate the square centered at (x, y) with the pair of integers $(i = nx, j = ny)$. Here $i, j \in \{0, 1, \ldots, n-1\}$. The transition probabilities for the corresponding Markov chain (whose elements are numbered as $jn + i + 1$) are shown in the following table:

	$i+j, i+2j$	$i+j, i+2j+1$	$i+j+1, i+2j+1$	$i+j, i+2j-1$	$i+j-1, i+2j-1$
i, j	1/2	1/8	1/8	1/8	1/8

Moduli of the "second" eigenvalues (r_2) of the transition matrices ($n^2 \times n^2$), and their multiplicities (in parenthesis) are

n	r_2	n	r_2	n	r_2	n	r_2	n	r_2
2	0.3968(3)	3	0.3953(8)	4	0.4543(12)	5	0.4029(20)	6	0.4443(24)
7	0.5577(16)	8	0.4940(24)	9	0.5038(24)	10	0.4754(12)	11	0.6203(20)
12	0.4567(48)	13	0.5371(28)	14	0.5577(16)	15	0.5495(16)	16	0.4940(24)
17	0.5864(36)	18	0.6733(24)	19	0.5976(36)	20	0.4902(24)	21	0.5838(32)

In this case all "second" eigenvalues are greater in modulus than $1/\Lambda$. Looking at these two tables the structure of limit points of the eigenvalues seems not quite clear for both of the considered families of the partitions, and one might argue that for large enough n the corresponding r_2 may converge to $1/\Lambda$. However, we have numerical evidence that for the standard partition for $n = 7k$ there is an eigenvalue 0.4886, while for the shifted standard partition for $n = 8k$ there is an eigenvalue 0.4940. The following general statement justifies this prediction and provides us with the precise description of the structure of the spectrum for the case of a linear automorphism preserving integer points.

Theorem 3.2 *Let $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map such that $\tilde{f}(\mathbb{Z}^d) = \mathbb{Z}^d$, and let the map $f := \tilde{f} \,(\mathrm{mod}\ 1)$ be defined on the d-dimensional unit torus. Denote by P_n the matrix corresponding to Ulam's approximation of the map f constructed according to the partition of the unit torus into n^d equal cubes. Then $r \in \Sigma(P_{kn})$ for any positive integer k whenever $r \in \Sigma(P_n)$.*

The proof of this result is based on the fact that due to the selfsimilar structure of Ulam's approximation in this case both the matrix P_{kn} and the eigenvector e_{kn} corresponding to the eigenvalue r consist of repeated blocks of the matrix P_n and the eigenvector e_n respectively.

Therefore if for some n we obtain numerically a "bad" value for the "second" eigenvalue, it will still be present for large enough multiples of n.

In recent papers [9, 7] it was proposed to use Ulam's procedure based on a finite Markov partition to estimate r_2. This claim was justified in these papers for 1D smooth expanding maps and 2D Anosov automorphisms. In practice, the usefulness of this approach is limited by the observation that usually such partitions can be found only numerically, and as we shall show a small error here may lead to even worse accuracy of the eigenvalues compared to a non Markov partition.

Now we are in a position to answer the question why the spectral gap in the above numerics differs significantly from theoretical predictions. To have a simple model for the analytical study to start with, consider a family of 2D maps from the unit square into itself, defined as follows: $f_\gamma(x, y) := (2x \,(\mathrm{mod}\ 1)\,, \gamma(y - c) + c \,(\mathrm{mod}\ 1)\,)$, $\gamma \geq 0$, $0 < c < 1$. For each value of γ the map f_γ is a direct product of two 1D maps. Thus, for $\gamma > 1$ one can prove that the PF-spectrum of this map is just the set of all pairwise products of elements of the spectra of the involved 1D maps. For $\gamma \leq 1$, however, only the first map contributes to the spectrum. Observe that for $\gamma < 1$ the invariant measure is concentrated on the attracting fiber $\Gamma := \{(x, c) : \ 0 \leq x \leq 1\}$, and the spectrum is the PF-spectrum of the piecewise expanding map on Γ. The dependence of r_2 on the parameter γ is shown by thick lines in Figure 4. Observe the discontinuous behaviour when the parameter γ crosses the value 1. By dots we indicate the rate of correlations decay with respect to the Lebesgue measure in this system. Observe that these two graphs differ only for $0 < \gamma < 1$, i.e. when the stable foliation is present.

This simple example demonstrates the main difference between expanding and hyperbolic maps, because the "traditional" spectrum in the latter case does not take into account the behaviour of the system along the stable foliation.

3.4 Random perturbations of contractive maps.

The above examples show that in order to understand how small random perturbations change the behaviour of a hyperbolic system one has to study their influence on a pure contractive

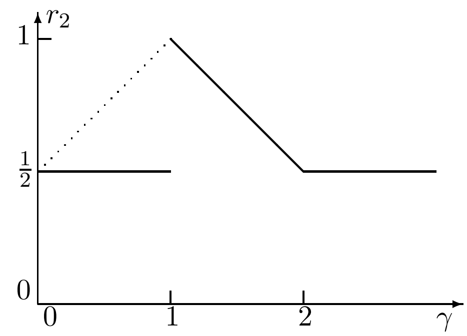

Figure 4: $r_2(\gamma)$ for the direct product map.

map. Let $f_{\gamma,c}(x) := \gamma(x-c)+c$, $0 < \gamma, c < 1$, be a family of maps from the unit interval $[0,1]$ into itself. If our random perturbation has the transition probability density $q(\cdot,\cdot)$, being a **BV** function of the first variable, then the corresponding transition operator is well defined as an operator from **BV** into itself and one can compute its spectrum. Denote by $[x]$ the closest integer to the point x, and let $\delta = |[cn] - cn| \in [0, 1/2]$ be the distance from the fixed point c to the closest end-point of the Ulam interval to which it belongs, multiplied by n. The transition matrix P_n is lower triangular in this case. Therefore the diagonal entries of the matrix are just its eigenvalues. A simple calculation gives the following representation for $\tilde{r}_2$ as a function of δ:

$$\tilde{r}_2(\delta) = \begin{cases} 2 - \frac{1}{\gamma} & \text{if } \delta = 0 \text{ and } \gamma > \frac{1}{2} \\ 1 - \delta(\frac{1}{\gamma} - 1) & \text{if } 0 < \delta \leq \frac{\gamma}{1-\gamma} \\ 0 & \text{otherwise.} \end{cases}$$

This result shows the following. First, $\tilde{r}_2$ sensitively depends on the distance to the closest end-point of the Ulam interval it belongs to: $\tilde{r}_2(0) = 2 - \frac{1}{\gamma}$, $\tilde{r}_2(0_+) = 1$, while $\tilde{r}_2(1/2) = (3 - 1/\gamma)/2$ (provided $\gamma > 1/2$). Second, when the fixed point lies very close to the boundary of one of the Ulam intervals, the estimate is the worst, which yelds a very bad accuracy if one uses an approximation to a Markov partition for Ulam's procedure.

Let us show that shift-invariant random perturbations may cure this pathology. Suppose the random perturbation has a shift-invariant transition probability density $q(x,y) = q(y-x)$, $q \in C^2$. We want to advocate that the PF-spectrum of the perturbed operator in zero noise limit in this case is well defined, does not depend on the shape of $q(\cdot)$ and is nontrivial. Let us prove this for $\tilde{r}_2$.

Lemma 3.1 *Let $q \in C^2$, $q(x) = 0$ if $|x| > \varepsilon$, and $\varepsilon \leq \max\{\gamma c, \gamma(1-c)\}$. Then $\tilde{r}_2 = \gamma$.*

Proof. The random map can be rewritten as $x_{n+1} = \gamma(x_n - c) + c + \xi_n$, where (ξ_n) is a sequence of iid random variables with probability density $q(\cdot)$. Therefore

$$x_{n+1} = \gamma^n(x_1 - c) + c + (\gamma^{n-1}\xi_1 + \gamma^{n-2}\xi_2 + \ldots + \xi_n).$$

Let $\xi^{(n)} := c + \sum_{k=0}^{n-1} \gamma^k \xi_{n-k}$. Since the ξ_k are iid, the sequence $(\xi^{(n)})$ converges in distribution to a random variable $\xi^{(\infty)}$. Then the random variables x_n converge in distribution to $\xi^{(\infty)}$

as $n \to \infty$ and $\tilde{r}_2$ corresponds to the rate of this convergence in the following sense: $\xi^{(\infty)}$ can be rewritten as $\xi^{(\infty)} \stackrel{\mathrm{d}}{=} \gamma^n(\tilde{\xi}^{(\infty)} - c) + \xi^{(n)}$, where $\stackrel{\mathrm{d}}{=}$ means equality in distribution and $\tilde{\xi}^{(\infty)}$ is a copy of $\xi^{(\infty)}$ which is independent of the $\xi^{(n)}$. Denote by q_n, q_∞, h_n, ϕ_n and ψ_n the densities of the random variables $\xi^{(n)}, \xi^{(\infty)}$, x_n, $\gamma^n(x_1 - c)$ and $\gamma^n(\xi^{(\infty)} - c)$ respectively. Then $h_{n+1} = q_n \star \phi_n$ and $q_\infty = q_n \star \psi_n$, and the supports of ϕ_n and ψ_n are of order γ^n. Observing that for any $h \in C^1$

$$\int |h(x+\delta) - h(x)|\, dx = (|\delta| + o(\delta))\mathrm{var}(h),$$

we conclude that for large n

$$||q_n - q_\infty||_1 = \int |q_n(x) - q_n(x-y)|\psi_n(y)\, dy\, dx \le O(\gamma^n)\mathrm{var}(q_n),$$

$$\mathrm{var}(q_n - q_\infty) = \int |q_n'(x) - q_n'(x-y)\psi_n(y)|\, dy\, dx \le O(\gamma^n)\mathrm{var}(q_n'),$$

and similarly

$$||q_n - h_{n+1}||_1 \le O(\gamma^n)\mathrm{var}(q_n), \qquad \mathrm{var}(q_n - h_{n+1}) \le O(\gamma^n)\mathrm{var}(q_n).$$

As $\xi^{(n)} = \xi_n + \gamma\xi^{(n-1)}$, there is some probability density h such that $q_n = q \star h$. Therefore $\mathrm{var}(q_n) \le \mathrm{var}(q)$ and $\mathrm{var}(q_n') \le \mathrm{var}(q')$ so that

$$||h_n - q_\infty||_{_{\mathrm{BV}}} \le O(\gamma^n) \cdot (\mathrm{var}(q) + \mathrm{var}(q')).$$

It is easily seen that one can choose the initial density h_1 such that this order of convergence is attained (e.g. take h_1 close to a δ-function). This yields the claim of the lemma. ∎

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